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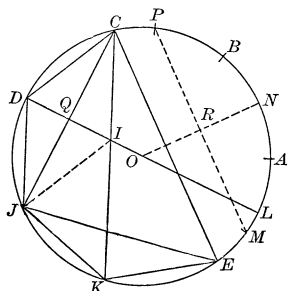
# CONCERNING THE REGULAR INSCRIBED HEPTAGON.

By S. A. JOFFE, New York City.

Mr. J. Q. McNatt's article "A Geometrical Discussion of the Regular Inscribed Heptagon," which appeared in the January number of this MONTHLY, pages 13 and 14, contains a very interesting and ingenious method of arriving at what the author calls "The Heptagon Cubic," from the solution of which he shows that the square of the side of the regular heptagon equals  $\frac{3}{4}$ , as a very close approximation.

Without detracting from the value of the method, I should like to point out that some of the details in the process may be omitted or simplified, and it has occurred to me that it may be worth while to present here the method in an abbreviated form, thus making the matter more widely known. In preparing the following lines, I have endeavored to preserve as much of the original figure and of the author's text as was consistent with the object of simplification.

To calculate the length of the side of a regular inscribed heptagon in terms of the radius as a unit, suppose, in the accompanying figure, that the circumference is divided into seven equal parts at the points *A, B, C, D, J, K*, and *E*.



Draw the diameter *DL* and the chords *JE*, *CJ*, *CK* and *CE*, and let *CJ* and *CK* cut *DL* at *Q* and *I* respectively. Join *I* and *J*.

Let the length of the side of the regular inscribed heptagon be *h* units. We have  $DL : DC :: DC : DQ$ ; hence  $DQ = \frac{1}{2}h^2$ . Moreover,  $QC = \sqrt{DC^2 - DQ^2} = \sqrt{h^2 - \frac{1}{4}h^4} = \frac{1}{2}h\sqrt{4 - h^2}$ . From the isosceles triangle *DCI* we have  $CI = CD = h$ , and  $DI = 2DQ = h^2$ ; from the isosceles triangle *CIJ* we have  $IJ = CI = h$ , and  $CJ = 2CQ = h\sqrt{4 - h^2}$ , so that

$$(1) \quad JE = h\sqrt{4 - h^2}.$$

Now, by geometry  $IL \times DI = CI \times IK$ , and since  $IL = DL - DI = 2 - h^2$ , we have  $(2 - h^2) \times h^2 = h \times IK$ ; hence

$$(2) \quad IK = 2h - h^3,$$

and since  $CK = CI + IK = 3h - h^3$ , and  $CE = CK$ , therefore

$$(3) \quad CE = 3h - h^3.$$

The two isosceles triangles  $CJE$  and  $IJK$ , having equal angles at  $E$  and  $K$  respectively, are similar. Hence we have  $JE : CE :: JK : IK$ , or, using (1), (2) and (3)

$$h\sqrt{4 - h^2} : (3h - h^3) :: h : (2h - h^3).$$

Hence

$$(4') \quad (2 - h^2)\sqrt{4 - h^2} = 3 - h^2.$$

Squaring, we get

$$(4 - 4h^2 + h^4)(4 - h^2) = 9 - 6h^2 + h^4,$$

and expanding,

$$16 - 20h^2 + 8h^4 - h^6 = 9 - 6h^2 + h^4.$$

Finally, transposing and simplifying, we obtain the author's *Heptagon Cubic*:

$$(5) \quad 7 - 14h^2 + 7h^4 - h^6 = 0.$$

Solving by Horner's method, we find  $h^2 = .7530203962821 \dots = \frac{3}{4}$ , approximately.

*Remark.* It will thus be seen that while there is introduced a new line  $IJ$ , we dispense with the consideration of the line  $OC$ , and with both the consideration and the computation of the author's lines  $SE$ ,  $SJ$ ,  $SK$  and  $SC$ . As a result, the equation (4') appears in a much simpler form than the author's equation (4).

The approximate construction of the heptagon may also be simplified as follows:

Let  $M$ ,  $N$  and  $P$  be three consecutive vertices of an inscribed regular hexagon. Draw the chord  $MP$  and the radius  $ON$ , and let  $MP$  meet  $ON$  in  $R$ . Then  $MR$  is, approximately, the length  $h$  of the side of the regular inscribed heptagon. The reason is self-evident: approximately,  $h = \frac{1}{2}\sqrt{3}$ , and  $MP$ , as the side of a regular inscribed triangle,  $= \sqrt{3}$ , so that  $MR = \frac{1}{2}\sqrt{3}$ , and therefore  $MR = h$ , approximately.

## A PROBLEM IN NUMBER THEORY.

By GEO. A. OSBORNE, Massachusetts Institute of Technology.

§ 1. When is the sum of the squares of two successive integers a perfect square? The following are examples:

$$3^2 + 4^2 = 5^2, \quad 20^2 + 21^2 = 29^2. \quad \text{The next is } 119^2 + 120^2 = 169^2.$$

The numbers 3, 20, 119, . . . are the terms of a series

$$0, 3, 20, 119, 696, \dots u_n, u_{n+1}, \quad (1)$$

where

$$u_{n+1} = 6u_n - u_{n-1} + 2. \quad (2)$$

This may be proved as follows: